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BRST algebra quantum double and quantization of the proper time cotangent bundle

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Abstract. The quantum double for the quantized BRST superalgebra is studied. The corresponding $\mathcal{R}$-matrix is explicitly constucted. The Hopf algebras of the double form an analytical variety with coordinates described by the canonical deformation parameters. This provides the possibility to construct the nontrivial quantization of the proper time supergroup cotangent bundle. The group-like classical limit for this quantization corresponds to the generic super Lie bialgebra of the double.

1. Introduction

The theories of Casalbuoni–Brink–Schwarz (CBS) superparticle [1] are fundamentally related to supersymmetric field theories and strings. Superparticle orbits are determined up to local fermionic (Siegel) transformations [2], which play a crucial role in removing the unphysical degrees of freedom. For the case of a superparticle it has been shown [3] that Siegel symmetry can be interpreted as the usual local proper-time supersymmetry (PTSA). The equivalence between CBS superparticle and the spinning particle was established [4] by identifying Lorentz-covariant Siegel generator with the local proper-time supersymmetry of the spinning particle [5].

To quantize such models it is natural to apply the BRST formalism, which is manifestly Lorentz-invariant. For the point-particle case the BRST quantization starts with the Faddeev–Popov prescription and the extraction of a new nilpotent symmetry operator. The latter can be included in the algebra ILI(1) [6].

Thus the symmetry algebra of a system with superparticles contain both BRST and PTSA subalgebras. The simplest possible unification of them is the direct sum. It is natural to consider the properties of quantum analogues of (PTSA) $\oplus$ (BRST). On the other hand BRST algebra itself can be treated as a deformation of the trivial algebra of coordinate functions for the superparticle. So one can equally consider q-deformations of a unification of PTSA with Abelian superalgebra creating the BRST subalgebra in the process of deformation. In this case the initial unification is a semidirect sum corresponding to the coadjoint action.

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The significant feature of the symmetries PTSA and BRST is that their superalgebras are dual. This gives the opportunity to obtain the necessary q-deformed symmetry by constructing a Drinfeld double for a quantized $(\mathrm{PTSA})_q$ superalgebra. The latter is easily obtained using the method developed in [7].

In this paper we demonstrate that the Hopf algebra of the quantum superdouble $SD(\mathrm{PTSA}_q, \mathrm{BRST}_q)$ can be treated as a quantized symmetry for both interpretation schemes presented above. For the first one the double must be considered as a quantum group corresponding to the algebra $(\mathrm{PTSA}) \oplus (\mathrm{BRST})^{\mathrm{opp}}$. In the second approach the multiplications in SD are treated as the deformed algebra of coadjoint extension of (PTSA).

The paper is organized as follows. In the second section all the necessary algebraic constructions are obtained including the explicit expression of the $\mathcal{R}$-matrix for $SD(PTAS_q, \mathrm{BRST}_q)$. In section 3 the dual canonical parameters are introduced in SD. This gives the possibility to construct the limit transitions connecting different Poisson structures in the created set of Hopf algebras. All the necessary classical limits are explicitly realized. The obtained results are discussed in section 3 from the point of view of possible physical interpretation.

2. The BRST algebra quantum double

Let the Hopf algebra with the generators $\{T, S\}$ and the defining relations

$$\begin{aligned}
&[T, S] = 0 \\
&\{S, S\} = 2\frac{\sinh(hT)}{\sinh(h)} \\
&\Delta T = T \otimes 1 + 1 \otimes T \\
&\Delta S = \mathrm{e}^{hT/2} \otimes S + S \otimes \mathrm{e}^{-hT/2}
\end{aligned} \tag{1}$$

be interpreted as the proper-time quantum superalgebra (PTSA_q). Choose the following quantization of the two-dimensional BRST algebra with basic elements $\{\tau, \xi\}$:

$$\begin{aligned}
&[\tau, \xi] = \frac{h}{2}\xi \\
&\{\xi, \xi\} = 0 \\
&\Delta\tau = \tau \otimes 1 + 1 \otimes \tau + \frac{h}{\sinh(h)}\xi \otimes \xi \\
&\Delta\xi = \xi \otimes 1 + 1 \otimes \xi.
\end{aligned} \tag{2}$$

Consider the generators τ and ξ as dual to T and S:

$$\begin{aligned}
&\langle \xi, S\rangle = 1 \qquad \langle \xi, T\rangle = 0 \\
&\langle \tau, S\rangle = 0 \qquad \langle \tau, T\rangle = 1.
\end{aligned}$$

This dualization induces a new Hopf algebra structure $(\mathrm{PTSA}_q)^*$ (with generators τ and ξ). The multiplication in BRST_q algebra (defined by (2)) is the same as in $(\mathrm{PTSA}_q)^*$ while the coproduct is opposite.

Note that according to the quantum duality principle [8, 10, 9] the PTSA_q algebra also defines the quantization of the two-dimensional vector quantum group described by the coproducts in (1). This is the semidirect product of two Abelian groups and its supergroup nature is reflected only by the fact that its topological space is a superspace. The quantum supergroup (different from the previous one) is also defined by the Hopf algebra BRST_q (see Δ's in (2)).

To obtain the quantum superdouble $SD(\mathrm{PTSA}_q, \mathrm{BRST}_q)$ one can start by constructing the corresponding universal element. Let us define the Poincare–Birkhoff–Witt (PBW) basis for PTSA_q and BRST_q:

$$\begin{aligned}&1, \xi, \frac{\tau^n}{n!}, \frac{\xi\tau^n}{n!}, \ldots\\ &1, S, \frac{T^n}{n!}, \frac{ST^n}{n!}, \ldots .\end{aligned} \tag{3}$$

The universal element can be written in the form

$$\mathcal{R} = (1 \otimes 1 + S \otimes \xi)\mathrm{e}^{T\otimes\tau}. \tag{4}$$

Its main properties are easily checked with the help of an auxiliary relation

$$\begin{aligned}&\left(1 \otimes 1 \otimes 1 + \frac{(\mathrm{e}^{2hT} - 1)}{\mathrm{e}^{h} - \mathrm{e}^{-h}} \otimes \xi \otimes \xi\right) \exp(T \otimes 1 \otimes \tau + T \otimes \tau \otimes 1)\\ &\quad = \exp\left(T \otimes \tau \otimes 1 + T \otimes 1 \otimes \tau + \frac{h}{\sinh(h)} T \otimes \xi \otimes \xi\right).\end{aligned} \tag{5}$$

The next step should involve the construction of the multiplication rules consistent with this $\mathcal{R}$-matrix. For any pair of dual Hopf algebras H and H^* with the basic elements $\{e_s\}$ and $\{e^t\}$ and the universal element $\mathcal{R} = e_s \otimes e^s$ the following relation is valid both for ordinary Hopf algebras as well as for super-Hopf ones:

$$(m \otimes \mathrm{id})[(1 \otimes \mathcal{R}_1 \otimes \mathcal{R}_2)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\mathrm{id} \otimes \mathrm{id} \otimes \boldsymbol{S}^{-1})(\Delta \otimes \mathrm{id})\Delta(e_s)] = (1 \otimes e_s)\mathcal{R}. \tag{6}$$

Let us rewrite the third defining relation, $\mathcal{R}\Delta(e) = \tau\Delta(e)\mathcal{R}$, in terms of structure constants,

$$(-1)^{\sigma_k\sigma_l+\sigma_k\sigma_j}\Delta_i^{kl} m_{lj}^t e_k e^j = (-1)^{\sigma_p\sigma_q}\Delta_i^{pl} m_{qp}^t e^q e_l. \tag{7}$$

Here $\sigma_k \equiv \sigma(k)$ is the grading function, m_{lj}^t (Δ_i^{kl}) are the structure constants for multiplication m (comultiplication Δ) in a Hopf algebra H with basis $\{e_s\}$. Below we shall also use the structure constants of compositions $e_n e_u e_k = m_{nuk}^t e_t$ and $(\Delta\otimes\mathrm{id})\Delta(e_s) = \mu_s^{klj} e_k \otimes e_l \otimes e_j$. From the formulae (6) and (7) the explicit form of multiplication rules follows:

$$e_s e^t = \sum_{n,l,k,u,j} (-1)^{\sigma_n(\sigma_l+\sigma_k)+\sigma_u\sigma_k+\sigma_s\sigma_t} m_{nuk}^t \mu_s^{klj} (\boldsymbol{S}^{-1})_j^n e^u e_l. \tag{8}$$

Despite the transparency of these rules it is not easy to use them directly. In close analogy with the case of the ordinary double some additional restructuring of the formula (8) is necessary. Calculate two similar expressions: one for the element e^t,

$$\Phi(e^t) \equiv (-1)^{\sigma_u\sigma_k} m_{nuk}^t e^n \otimes e^k \otimes e^u \tag{9}$$

the other for e_s,

$$\Psi(e_s) \equiv (\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\mathrm{id} \otimes \mathrm{id} \otimes \boldsymbol{S}^{-1}) \,\Box\, (e_s) = (-1)^{\sigma_l\sigma_j+\sigma_k\sigma_j} \Box_s^{klj} (\boldsymbol{S}^{-1})_j^n e_n \otimes e_k \otimes e_l \tag{10}$$

with $\Box \equiv \mu(\mu\otimes\mathrm{id})$, where μ is the multiplication in the dual Lie superalgebra (BRST_q in our case). To write down the product $e_s \cdot e^t$ it is sufficient to contract the first and the second tensor factors and to multiply the third ones:

$$(-1)^{\sigma_s\sigma_t} e_s \cdot e^t = \langle\Phi'(e^t), \Phi'(e_s)\rangle\langle\Phi''(e^t), \Phi''(e_s)\rangle\Phi'''(e^t) \cdot \Phi'''(e_s). \tag{11}$$

Applying these formulae to the pair $(\mathrm{PTSA}_q, \mathrm{BRST}_q)$ we obtain the Hopf superalgebra $SD(\mathrm{PTSA}_q, \mathrm{BRST}_q) \equiv SD^h$ with the defining relations:

$$
\begin{aligned}
&[T, S] = 0 \\
&[\tau, \xi] = \frac{h}{2}\xi && \{S, S\} = 2\frac{\sinh(hT)}{\sinh(h)} \\
&[S, \tau] = hs - 2\frac{h\xi}{\sinh(h)}\cosh\left(\frac{1}{2}hT\right) && \{\xi, \xi\} = 0 \\
&[T, \tau] = 0 && \{S, \xi\} = 2\sinh(\tfrac{1}{2}hT) \\
&[T, \xi] = 0
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\Delta T = T \otimes 1 + 1 \otimes T \\
&\Delta\xi = \xi \otimes 1 + 1 \otimes \xi \\
&\Delta S = \mathrm{e}^{\frac{hT}{2}} \otimes S + S \otimes \mathrm{e}^{-\frac{hT}{2}} \\
&\Delta\tau = \tau \otimes 1 + 1 \otimes \tau + \frac{h}{\sinh(h)}\xi \otimes \xi
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\boldsymbol{S}(T) = -T \qquad \boldsymbol{S}(\tau) = -\tau \\
&\boldsymbol{S}(S) = -S \qquad \boldsymbol{S}(\xi) = -\xi.
\end{aligned}
\tag{14}
$$

It is easy to check that the universal $\mathcal{R}$-matrix (4) realize the triangularity of this quantum superdouble.

3. Deformations of super Lie–Poisson (SL–P) structures induced by superdouble

The main aim of this section is to show that the superdouble SD not only provides the nontrivial unification of BRST and PTSA algebras but also induces the continuous (in fact analytical) paths connecting different Lie–Poisson (L–P) supergroups. We shall demonstrate that there are natural parameters that describe these transitions. The mechanical systems that can be attached to such L–P structures are close and one can use the above mentioned parameters to connect them. We shall also show that the analytical properties of such transitions make it possible to find new quantizations of the cotangent bundles of PTSA and BRST supergroups.

3.1. Analyticity and duality

Consider the variety $\mathcal{H}$ of Hopf algebras admitting a PBW basis of ordered monomials with a common finite assembly of generators $\{\mathbf{1}, e_1, e_2, \ldots, e_n\}$. Let L be the linear span of $\{e_1, e_2, \ldots, e_n\}$. For any fixed Hopf algebra $H \in \mathcal{H}$ and positive number l consider the space

$$M = \oplus_{k=0}^{l} L^{\otimes k}$$

and restrictions of compositions

$$m^H_{\downarrow M\otimes M\longrightarrow M} \qquad \Delta^H_{\downarrow M\longrightarrow M\otimes M} \qquad S^H_{\downarrow M\longrightarrow M} \qquad \eta^H_{\downarrow M\longrightarrow M} \qquad \epsilon^H_{\downarrow M\longrightarrow M}. \tag{15}$$

For each l the restricted list $C_{\downarrow}^{(H;l)}$ of structure constants for compositions (15) can be treated as a point in the corresponding finite-dimensional space $Y^{(l)}$. The variety $\mathcal{H}$ is called analytical if for every finite l the set $\{C_{\downarrow}^{(H;l)}\}_{H\in\mathcal{H}}$ forms an analytical variety in the space $Y^{(l)}$ [12].

For any finite-dimensional Lie algebra A the deformation quantization $U_h(A)$ of the universal enveloping algebra $U(A)$ is an example of one-dimensional analytical variety of Hopf algebras. In this case analyticity is mutually connected with the notion of classical limit.

The quantum duality principle asserts that quantization of a Lie bialgebra (A, A^*) gives rise to a dual pair of Hopf algebras $(U_p(A), U_p(A^*))$ or $(\mathrm{Fun}_p(G), \mathrm{Fun}_p(G^*))$ [8, 10]. Here G and G^* are the universal covering groups for A and A^* respectively. Thus each quantum algebra of this type can be interpreted as a quantum group of the dual simply connected group G^*,

$$U_p(A) \approx (\mathrm{Fun}_p(G^*))$$

and vice versa. This means that one and the same Hopf algebra (a deformation quantization of a L–P structure (A, A^*)) must have two different classical limits ($U(A)$ and $\mathrm{Fun}(G^*)$). The notion of the second classical limit was first introduced by Drinfeld [9]. It was shown [11] that the parameter p corresponding to the second classical limit can be introduced such that the subset $\{H_p\}$ forms an analytical subvariety with the limiting point $H_0 = \mathrm{Fun}(G^*)$. Thus the canonical form of deformation quantization of a Lie bialgebra (A, A^*) must form a two-parametric set $\mathcal{D}_{hp}$ of Hopf algebras $U_{hp}(A)$ (or $\mathrm{Fun}_p(G^*)$). This set is an analytical subvariety in $\mathcal{H}$. h and p are called dual canonical parameters for a quantized algebra (or group) [11].

3.2. Analytical subvariety of quantum doubles

Smooth parametrization of a deformation quantization can be inherited by constructions such as quantum doubles and crossed products [13]. It is easily seen that the set of quantum doubles $\{D(H_{hp}, H^*_{hp})\}$ is a two-parametric analytical subvariety in a (new) set of Hopf algebras with generators $\{\mathbf{1}, e^s \otimes e_t\}_{s,t=1,\ldots n}$. Note that in this case the distinguished points $h = 0$ or $p = 0$ do not correspond to classical limits of $D(H_{hp}, H^*_{hp})$.

Let us use the canonical parametrization of the Hopf algebra PTSA_q to obtain the two-dimensional analytical variety of quantum superdoubles SD^{hp} and extract the appropriate one-dimensional curve $SD^{(h)} \subset SD^{hp}$.

Applying quantum duality to the algebra PTSA_q one can introduce the canonical parameter p dual to h. The composition

$$\{s, s\} = 2p\frac{\sinh(hT)}{\sinh(h)}$$

is the only relation that changes. In the $(\mathrm{BRST})_q$ algebra the coproduct $\Delta(\tau)$ also acquires this dual parameter:

$$\Delta\tau = \tau \otimes 1 + 1 \otimes \tau + \frac{hp}{\sinh(h)}\xi \otimes \xi$$

(cf (2)). As a result we obtain the two-parametric family $SD^{hp}(\mathrm{PTSA}, \mathrm{BRST})$ of quantum doubles. It can be observed that in the Hopf algebra (12)–(14) the composition $[\tau, \xi]$ allows the rescaling

$$[\tau, \xi] = \tfrac{1}{2}\alpha h\xi$$

with the additional arbitrary parameter α. We shall consider the case $\alpha = 2$ (in order to have the necessary classical limits) and choose the one-dimensional family of Hopf algebras putting $p = 1 - h$. The defining relations for $SD^{h,1-h}_{\alpha=2} \equiv SD^{(h)}$ are

$$\begin{aligned}
&[\tau, \xi] = h\xi \\
&\{S, S\} = 2(1-h)\frac{\sinh(hT)}{\sinh(h)} \\
&\{S, \xi\} = 2\sinh\left(\frac{hT}{2}\right) \\
&[S, \tau] = hS - \frac{2h(1-h)}{\sinh(h)}\xi\cosh\left(\frac{hT}{2}\right) \\
&\Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau + \frac{h(1-h)}{\sinh(h)}\xi \otimes \xi \\
&\Delta(S) = \exp(\tfrac{1}{2}hT) \otimes S + S \otimes \exp(-\tfrac{1}{2}hT)
\end{aligned} \tag{16}$$

(from here on we expose only nonzero (super)commutators, nonprimitive coproducts and omit the antipodes (14)).

3.3. Smooth domain $\mathcal{D}^{(h)}_{\mu\theta}$ and SL–P structures

Let us show that the obtained curve can be used as a base for the fibre bundle whose leaves are the two-dimensional analytical varieties with (new) canonical parameters (μ, θ). As a result we shall obtain a three-dimensional analytical domain $\mathcal{D}^{(h)}_{\mu\theta}$ containing quantizations of different L–P structures and find pairs of them that are close to each other.

The easiest way to construct $\mathcal{D}^{(h)}_{\mu\theta}$ is to start with the two-dimensional leaves $(\mathcal{D}_{\mu\theta})$ referring to the boundary points $H^{(0)}$ and $H^{(1)}$ of $SD^{(h)}$ (see figures 1 and 2). It is not difficult to fix the vector field $\mathcal{F}^{(0)}_{\mu\theta}$ normal to that of these leaves and tangent to the curve $SD^{(h)}$. Then the internal part of $\mathcal{D}^{(h)}_{\mu\theta}$ is formed by the set of solutions of the corresponding differential equation (see figure 3).

According to the general theory of quantum double [10] the elements of the set $SD^{(h)}$ can be presented as deformation quantizations, the corresponding Lie superbialgebra can

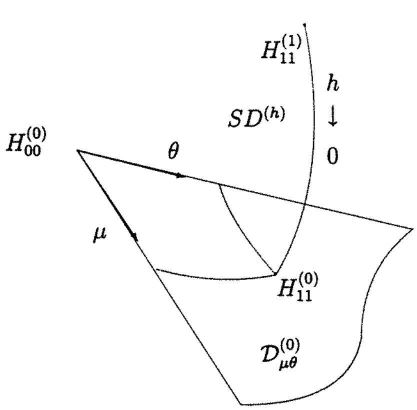

Figure 1.

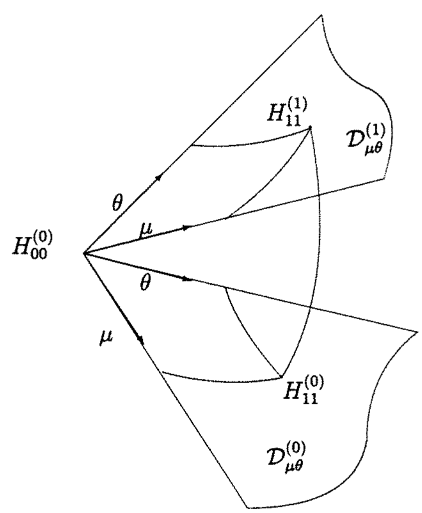

Figure 2.

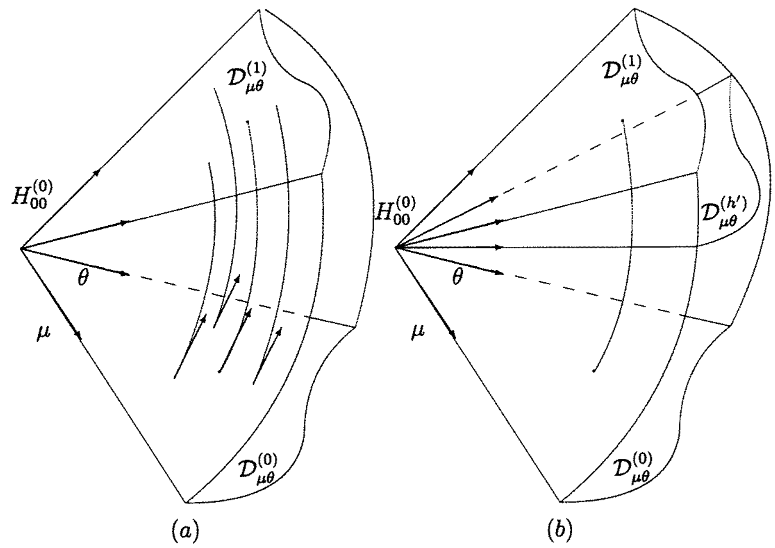

Figure 3.

be constructed using the classical Manin triple. Consider the Hopf algebra $H^{(0)} \in SD^{(h)}$ described by the relations (16) in the limit $h \to 0$:

$$\begin{aligned} &[S, \tau] = -2\xi \\ &\{S, S\} = 2T \end{aligned} \tag{17}$$

$$\Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau + \xi \otimes \xi. \tag{18}$$

This limit can be interpreted as a quantized semidirect sum $(\text{PTSA} \rhd \text{Ab})_q$ (Ab denotes the Abelian two-dimensional superalgebra). The corresponding analytical variety $\mathcal{D}^{(0)}_{\mu\theta}$ of Hopf algebras (see figure 1) is defined by the compositions

$$\begin{aligned} &[S, \tau] = -2\mu\xi \\ &\{S, S\} = 2\mu T \\ &\Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau + \theta\xi \otimes \xi. \end{aligned} \tag{19}$$

These relations correspond to the quantized SL–P structure in which the cocommutative superalgebra (PTSA $\rhd$ Ab) is deformed in the direction of the Poisson bracket $\{\xi, \xi\} = \tau\theta$. This quantization looks trivial, the multiplications in (19) do not depend on θ.

In the opposite limit $h \to 1$ the Hopf algebra $H^{(1)} \in SD^{(h)}$ presents a nontrivial deformation of a semidirect sum (BRST $\rhd$ Ab):

$$\begin{aligned} &[\tau, \xi] = \xi \\ &[S, \tau] = +S \\ &\{S, \xi\} = 2\sinh\left(\frac{T}{2}\right) \end{aligned} \tag{20}$$

$$\Delta(S) = \exp(\tfrac{1}{2}T) \otimes S + S \otimes \exp(-\tfrac{1}{2}T). \tag{21}$$

The procedure analogous to that used for $H^{(0)}$ leads to the analytical variety $\mathcal{D}^{(1)}_{\mu\theta}$ (see figure 2) of Hopf algebras

$$
\begin{aligned}
&[S, \tau] = +\mu S \\
&[\tau, \xi] = \mu\xi \\
&\{S, \xi\} = 2\frac{\mu}{\theta}\sinh\left(\frac{\theta T}{2}\right) \\
&\Delta(S) = \exp(\tfrac{1}{2}\theta T)\otimes S + S\otimes\exp(-\tfrac{1}{2}\theta T).
\end{aligned}
\tag{22}
$$

They have dual classical limits with the corresponding canonical parameters μ and θ. The two varieties $\mathcal{D}^{(0)}_{\mu\theta}$ and $\mathcal{D}^{(1)}_{\mu\theta}$ intersect in the trivial point—the Abelian and co-Abelian Hopf algebra $H^{(0)}_{00} = H^{(1)}_{00}$.

Let us show that there exists the continuous deformation [12] of the SP–L structure $\mathcal{D}^{(0)}_{\mu\theta}$ in the direction of $\mathcal{D}^{(1)}_{\mu\theta}$. The first-order deforming functions for such a deformation is a field on $\mathcal{D}^{(0)}_{\mu\theta}$ tangent to the flow connecting $\mathcal{D}^{(0)}_{\mu\theta}$ and $\mathcal{D}^{(1)}_{\mu\theta}$. Evaluating the difference between the compositions (22) and (19), comparing it with the curve (16) as a representative of the flow (see figure 3) we obtain the deforming field $\mathcal{F}^{(0)}_{\mu\theta}$:

$$
\begin{aligned}
&[S, \tau] = +\mu S + 2\mu\xi \\
&[\tau, \xi] = \mu\xi \\
&\{S, S\} = -2\mu T \\
&\{S, \xi\} = \mu T \\
&\Delta(S) = \frac{1}{2}\theta T\wedge \\
&\Delta(\tau) = -\theta\xi\otimes\xi.
\end{aligned}
\tag{23}
$$

One can integrate the equations

$$
\frac{\partial H^{(h)}_{\mu,\theta}}{\partial h}_{|h=0} = \mathcal{F}^{(0)}_{\mu\theta}
$$

imposing the boundary conditions $H^{(0)}_{\mu,\theta} \in \mathcal{D}^{(0)}_{\mu\theta}$, $H^{(1)}_{\mu,\theta} \in \mathcal{D}^{(1)}_{\mu\theta}$, and $H^{(h)}_{1,1} = SD^{(h)}$. One of the possible solutions is the three-dimensional variety $\mathcal{D}^{(h)}_{\mu\theta}$ of Hopf algebras with compositions

$$
\begin{aligned}
&[S, \tau] = +\mu h S - \mu\frac{2h(1-h)}{\sinh(h)}\xi\cosh\left(\frac{1}{2}h\theta T\right) \\
&[\tau, \xi] = \mu h\xi \\
&\{S, S\} = 2\frac{\mu}{\theta}(1-h)\frac{\sinh(h\theta T)}{\sinh(h)} \\
&\{S, \xi\} = 2\frac{\mu}{\theta}\sinh\left(\frac{1}{2}h\theta T\right) \\
&\Delta(S) = \exp(\tfrac{1}{2}h\theta T)\otimes S + S\otimes\exp(-\tfrac{1}{2}h\theta T) \\
&\Delta(\tau) = \tau\otimes 1 + 1\otimes\tau + \frac{h(1-h)}{\sinh(h)}\theta\xi\otimes\xi.
\end{aligned}
\tag{24}
$$

For each $h' \in [0, 1]$ fixed the two-dimensional subvariety $\mathcal{D}^{(h')}_{\mu\theta}$ defines the SL–P structure,

$$\begin{aligned}
&[S,\tau] = +\mu h' S - \mu \frac{2h'(1-h')}{\sinh(h')}\xi \\
&[\tau,\xi] = \mu h'\xi \\
&\{S,S\} = 2\mu(1-h')\frac{h'}{\sinh(h')}T \\
&\{S,\xi\} = \mu h' T
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\delta(S) = \tfrac{1}{2}h'\theta T \wedge S \\
&\delta(\tau) = \frac{h'(1-h')}{\sinh(h')}\theta\xi \otimes \xi
\end{aligned} \tag{26}$$

described here as a pair of superalgebra (25) and supercoalgebra (26). For $h' \in (0, 1)$ these structures are equivalent. However, this is not true for the limit points—$\mathcal{D}^{(0)}_{\mu\theta}$ and $\mathcal{D}^{(1)}_{\mu\theta}$ represent two different contractions of the quantized SL–P structure $\mathcal{D}^{(h')}_{\mu\theta}|_{h'\in(0,1)}$ (see figure 3).

3.4. New forms of deformation quantizations

The results of the previous section show that any quantized algebra $H^{(h')}_{\mu,\theta}$ can be treated as being close to $H^{(0)}_{\mu,\theta}$ (or to $H^{(1)}_{\mu,\theta}$). The same will be true for the corresponding quantum dynamical systems. This is the quantum analogue of a well known property: the symmetry described by the classical double $d(A, A^*)$ is close to that of the semidirect sum $A \triangleright \mathrm{Ab}$ (or $A^* \triangleright \mathrm{Ab}$). We have demonstrated above that each point $H^{(h)}_{\mu,\theta}$ belongs to a fixed leave $\mathcal{D}^{(h)}$ and thus is a quantization of a unique SL–P structure (25), (26). Now we shall show that due to the analyticity of $\mathcal{D}$ each internal point $H^{(h')}_{\mu,\theta}$ can also be incorporated in a two-dimensional analytical subvariety $\tilde{\mathcal{D}}$ corresponding to other SL–P structures (whose quantizations belong to the domain $\mathcal{D}$). We shall prove this by presenting the explicit form of a subvariety $\tilde{\mathcal{D}}_{\mu\theta}$ (see figure 4) that describes the quantizations of the semidirect sum (PTSA $\triangleright$ Ab):

$$\begin{aligned}
&[S,\tau] = +\mu h S - \mu \frac{2h(1-h)}{\sinh(h)}\xi \cosh\left(\frac{1}{2}h^2 T\right); \\
&[\tau,\xi] = \mu h\xi \\
&\{S,S\} = 2\frac{\mu}{h}(1-h)\frac{\sinh(h^2 T)}{\sinh(h)} \\
&\{S,\xi\} = 2\frac{\mu}{h}\sinh\left(\frac{1}{2}h^2 T\right) \\
&\Delta(S) = \exp(\tfrac{1}{2}h^2 T) \otimes S + S \otimes \exp(-\tfrac{1}{2}h^2 T) \\
&\Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau + \frac{h^2(1-h)}{\sinh(h)}\xi \otimes \xi.
\end{aligned} \tag{27}$$

Thus the main statement is illustrated: the SL–P structure (17), (18) ('trivially' quantized as $\mathcal{D}^{(0)}_{\mu\theta}$) can be deformed in the direction of Hopf algebras belonging to $\mathcal{D}^{(1)}_{\mu\theta}$ (that is, by the field $\mathcal{F}^{(0)}_{\mu\theta}$). One of the classical limits (for $\mu \to 0$) belongs to the facet $\mathcal{D}^{(h)}_{0\theta}$ of classical supergroups (13). Note that despite these properties the Hopf algebra (27) is a quantization of the same super Lie bialgebra as in the trivial canonical quantization of the proper time

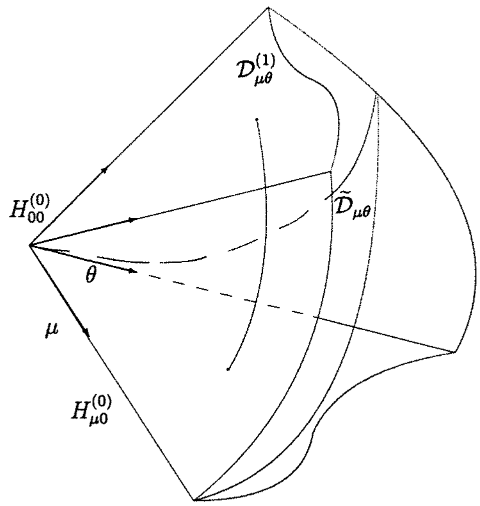

Figure 4.

group cotangent bundle (19). This is easily checked by evaluating the first-order terms in the expansion of the compositions (27) with respect to μ and h. This deformation is induced by the quantum superdouble construction.

Note. Earlier (see [12]) it was demonstrated that quantum doubles could induce even more complicated deformations of L–P structures where the corresponding groups and algebras of observables are not only deformed but also quantized. In the case discussed above the procedure presented in [12] does not lead to nontrivial results The variety $\mathcal{D}^{(0)}_{\mu\theta}$ lifted in the domain of non(anti)commutative and nonco(anti)commutative Hopf algebras will have edges equivalent to its internal points. This is a consequence of the equivalence of all the Hopf algebras corresponding to the internal points of $\mathcal{D}^{(h)}_{\mu\theta}$.

4. Conclusions

Analyticity plays an important role in the selection of admissible transformations of Poisson structures. Although the SL–P structures corresponding to $\{\mathcal{D}^{(h')}_{\mu\theta} | h' \in (0, 1)\}$ are equivalent, the continuous 'rotation' of $\mathcal{D}^{(h')}_{\mu\theta}$ breaks the analyticity. This is in accordance with the fact that the compositions (25), (26) with different h''s do not form super Lie bialgebras. This effect was first observed in [13] for a nonsuper case.

The deformations $\mathcal{D}^{(0)}_{\mu\theta} \longrightarrow \mathcal{D}^{(h')}_{\mu\theta}$ might be of considerable physical importance. Analyticity of $\mathcal{D}^{(0)}_{\mu\theta}$ means that the system with the four-dimensional phase superspace (the space of the group $\mathcal{AB} \times (\mathrm{BRST})^*$) can be quantized with the Poisson algebra (PTSA $\triangleright$ AB) (see figure 1) so that the multiplication of $\mathcal{AB} \times (\mathcal{BRST})^*$ is the Poisson map. Similarly, the existence of an analytical variety $\mathcal{D}^{(h')}_{\mu\theta}$ signifies that on the same superspace (now considered as the space of $(\mathcal{PTSA})^* \times (\mathcal{BRST})^*$) the quantization of the Poisson algebra equivalent to classical double of PTSA and BRST can be performed (and correlated with the group composition). The deformation $\mathcal{D}^{(0)}_{\mu\theta} \longrightarrow \mathcal{D}^{(h')}_{\mu\theta}$ analytically connects these two possible dynamics so that all the parameters of the first system can be considered as close to those of the second. We would like to stress that in these deformations both the supergroup and the Poisson superalgebra of its coordinate functions are deformed simultaneously. Such a process cannot be subdivided into successive deformations of group and algebra for the reasons described above. Thus the deformation of the dynamics must be accompanied by the deformation of the symmetry of the space.

The second valuable property of analytical domains such as $\mathcal{D}$ is the possibility to construct more complicated quantum deformations of the 'boundary' SL–P structures described by $\mathcal{D}^{(0)}$ and $\mathcal{D}^{(0)}$. This property does not depend on the peculiarities of superalgebras considered above. For any Lie bialgebra (A, A^*) the system with the phase space described by the cotangent bundle $T^*(G^*)$ and with the Poisson algebra $(A \triangleright \mathrm{Ab})$ can be quantized not only as the quantum double $D((A \triangleright \mathrm{Ab}), (\mathrm{Ab} \oplus A))$ (the lower subvariety $\mathcal{D}^{(0)}_{\mu\nu}$) but also to generate the Hopf algebras $D(d(A, A^*), (A^* \oplus A))$ (where d is the classical double). This property doesn't contradict the previous conclusion about the simultaneity of possible deformations in $\mathcal{D}$. One can easily check that the deformation $(A \triangleright \mathrm{Ab}) \longrightarrow d(A, A^*)$ must be accompanied here by the deformation $(\mathrm{Ab} \oplus A) \longrightarrow (A^* \oplus A)$. In our particular case Hopf algebras $H^{(h')}_{\mu,\theta}$ belonging to $\tilde{\mathcal{D}}$ can be treated as quantizations of $\mathrm{Fun}(\mathcal{AB} \times (\mathcal{BRST})^*)$ with the Poisson brackets generated by $(\mathrm{PTSA} \triangleright \mathrm{Ab})$ and the deformations of algebraic and coalgebraic parts are tightly correlated (see(27)).

We do not discuss here the details of quantized dynamical systems mentioned above. Our aim was to demonstrate the possibilities of the 'analytical' approach in the case where the supersymmetry is essential and we have chosen the simplest example for this purpose. To obtain physically meaningful scheme one must use more complicated constructions such as Yangian superdoubles [14]. It is clear that the tools used in the above treatment of BRST algebra quantum doubles can also be applied in the case of infinitely generated Hopf algebras (such as quantum affine algebras).

It should be mentioned that other methods of symmetry unification such as crossproducts or cocyclic cross- and bicrossproducts of Hopf algebras do not lead to nontrivial algebraic constructions in the case of PTSA_q and BRST_q.

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